On zero-curvature representations of evolution equations

1995 J. Phys. A: Math. Gen. 28 2861

(http://iopscience.iop.org/0305-4470/28/10/016)

J. Phys. A: Math. Gen. 28 (1995) 2861–2869. Printed in the UK

On zero-curvature representations of evolution equations

S Yu Sakovich

Institute of Physics, Academy of Sciences, Minsk 72, Belarus

Received 4 August 1994, in final form 3 January 1995

Abstract. For zero-curvature representations (ZCRs) $A_t - B_x - AB + BA = 0$ of evolution equations $u_t = f(x, u, u_x, \ldots, u_{x\ldots x})$, we develop a description which is invariant under gauge transformations $A' = SAS^{-1} - S_x S^{-1}$ and $B' = SBS^{-1} - S_t S^{-1}$, where A, B and S are matrix functions of $x, u, u_x, u_{xx}, \ldots$. We prove that every fixed matrix A of any dimension and order (in $u_{x\ldots x}$) determines a continual class of evolution equations which admit ZCRs with this A. Then we quote examples illustrating how a dependence of A on an essential parameter restricts classes of represented equations. One of our examples shows that some non-integrable systems can admit parametric Lax pairs and infinitely many non-trivial conservation laws.

Many remarkable nonlinear partial differential equations of modern mathematical physics can be represented as the compatibility condition

$$Z \equiv D_t A - D_x B - [A, B] = 0 \tag{1}$$

for the over-determined system of two linear equations

$$(D_x + A)\Phi = 0 \qquad (D_t + B)\Phi = 0 \tag{2}$$

where D_x and D_t are the total derivatives with respect to x and t, A and B are $m \times m$ matrix functions of independent variables x and t, dependent variables $u(x, t)$ and finite-order derivatives of u, $\Phi(x, t)$ is an m-component column, and square brackets denote the commutator [1–3]. Condition (1), often referred to as the *zero-curvature representation* (ZCR) due to its geometric interpretation [1], is said to represent an equation in u if all solutions u of the equation satisfy (1). The *gauge transformation* [1] of matrices A and B, namely,

$$A' = SAS^{-1} - (D_x S)S^{-1} \qquad B' = SBS^{-1} - (D_t S)S^{-1} \tag{3}$$

where S is any $m \times m$ matrix function of x, t, u and derivatives of u, $\det S \neq 0$, causes the tensor transformation $Z' = SZS^{-1}$ of ZCR (1); therefore two ZCRs, related by (3), are considered as equivalent. In soliton theory, where ZCR (1) appears as one of modifications of the Lax representation, it is very important that the auxiliary linear problem (2) contains a parameter in matrices A and B [1–3]. Only 'integrable' equations are believed to admit ZCRs with an *essential* ('spectral') parameter which cannot be removed ('gauged out') by gauge transformations (3) [4]. Though we use the traditional terms 'ZCR' and 'gauge transformations' in this paper, our approach to objects (1) and (3) will have no connection with differential geometry and gauge field theory: it will be based on an analogy between ZCRs and conservation laws.

Recently, two strange but similar results on ZCRs appeared in [5, 6]. Rabelo and Tenenblat [5] studied which of the evolution equations $u_t = u_{xxx} + f(u, u_x, u_{xx})$ admit ZCRs by 2×2 traceless matrices with A of a special form and found (besides known integrable equations) that the continual class of equations

$$u_t = D_x^2 p - D_x(up) \tag{4}$$

with arbitrary q in $p = u_x + q(u)$ admits ZCR (1) by

$$A = \frac{1}{2}\begin{pmatrix} \alpha & u+\alpha \\ u-\alpha & -\alpha \end{pmatrix} \qquad B = \frac{1}{2}\begin{pmatrix} -\alpha p & (D_x - u - \alpha)p \\ (D_x - u + \alpha)p & \alpha p \end{pmatrix} \tag{5}$$

where α is any constant. Evidently, this ZCR remains valid when we replace p in (4) and (5) by any function $p(x, u, \ldots, u_{x\ldots x})$. Alexeyev and Kudryashov [6] studied non-abelian pseudopotentials of evolution equation $u_t = (u^2)_{xx} + u - u^2$ and found that (1) with

$$A = \begin{pmatrix} \alpha & 0 \\ u & \alpha \pm 1 \end{pmatrix} \qquad B = \begin{pmatrix} \beta + \varphi & 0 \\ p & \beta \end{pmatrix} \tag{6}$$

represents the continual class of equations

$$u_t = D_x p \pm p + \varphi u \tag{7}$$

where function $p(x, u, \ldots, u_{x\ldots x})$ and constants α, β and φ are arbitrary. *Continual* classes (4) and (7) are much wider than habitual *discrete* hierarchies of integrable equations. Moreover, when A contains derivatives, the following may happen: matrices

$$A = u_x\begin{pmatrix} 1 & 0 \\ \alpha \exp u & -1 \end{pmatrix} \qquad B = \begin{pmatrix} f - \alpha\beta \exp(-u) & \beta \exp(-2u) \\ \alpha f \exp u - \alpha^2 \beta & -f + \alpha\beta \exp(-u) \end{pmatrix} \tag{8}$$

where function $f(x, u, \ldots, u_{x\ldots x})$ and constants α and β are arbitrary, represent any equation $u_t = f(x, u, \ldots u_{x\ldots x})$.

In this paper, we will develop a description, *invariant* under gauge transformations (3), for ZCRs of evolution equations. We will prove that *every* matrix $A(x, u, \ldots, u_{x\ldots x})$, which contains *no* essential parameter, determines ZCR (1) for a continual class like (4) or (7), whereas 'pathological' ZCRs like (8) arise due to equivalence (3) between A and $A' = 0$. Then we will quote three examples illustrating three different structures of classes of equations represented by (1) *with* an essential parameter in A. One of our examples will show that some *non-integrable* systems *can* admit ZCRs with an essential parameter and possess infinitely many non-trivial conservation laws.

There are many similarities between ZCRs and conservation laws [7]. On the analogy of adding trivial conservation laws of the first kind [8], we can replace in (1) pair (A, B) by pair $(A + A_0, B + B_0)$ with any matrices A_0 and B_0 such that $A_0 = 0$ and $B_0 = 0$ for all solutions of the represented equation, thus obtaining an equivalent ZCR. (Equivalence (3) resembles adding trivial conservation laws of the second kind [8].) Therefore, studying ZCRs of evolution equations, we can choose A and B to be matrix functions of $t, x, u, u_x, \ldots, u_{x\ldots x}$ only. Any explicit dependence of A on t can be treated as a special case of ZCRs for systems of evolution equations by introducing additional dependent variable v, $v_t = 1$. Hence, we do not lose generality, considering ZCR (1) of evolution equation

$$u_t = f(x, u, u_1, \ldots, u_n) \tag{9}$$

by $m \times m$ matrices $A(x, u, u_1, \ldots, u_i)$ and $B(x, u, u_1, \ldots, u_{i+n-1})$, $u_k = \partial^k u/\partial x^k$, $k = 0, 1, 2, \ldots$, $u_0 = u$ (possible dependence of f and B on t can easily be restored in our final results). In the set of all such A and B, gauge transformations are defined by (3) with $S = S(x, u, \ldots, u_j)$, $D_x = \partial/\partial x + \Sigma_{k=0}^{\infty} u_{k+1}\partial_k$ and $D_t = \Sigma_{k=0}^{\infty}(D_x^k f)\partial_k$, where $\partial_k = \partial/\partial u_k$. ZCR (1) is admitted by (9) if, and only if, A, B and f satisfy the condition

$$\sum_{k=0}^{i} (D_x^k f)\partial_k A = \nabla B \tag{10}$$

where operator ∇ is defined as $\nabla M = D_x M + [A, M]$ for any matrix M. Note that (10) must be an *identity* in u, not an ordinary differential equation restricting solutions of (9), therefore we can consider variables $x, u, u_1, u_2, \ldots$ as mutually independent and thus treat ZCRs of evolution equations algebraically. Using identity $(D_x d)M = \nabla(dM) - d\nabla M$ valid for any function d and matrix M, we can rewrite (10) as

$$fC = \nabla P \tag{11}$$

where

$$C = \sum_{k=0}^{i} (-\nabla)^k (\partial_k A) \tag{12}$$

$$P = B - \sum_{k=1}^{i} \sum_{l=1}^{k} (D_x^{k-l} f)(-\nabla)^{l-1}(\partial_k A). \tag{13}$$

On the analogy of conservation laws [8], we will refer to (11) as the *characteristic form* of ZCR (1) and to matrix C (12) as the *characteristic* of A.

Theorem 1. Characteristic C and matrix P are tensors under gauge transformations (3), $C' = SCS^{-1}$ and $P' = SPS^{-1}$. Characteristic C is zero if, and only if, matrix $S(x, u, \ldots, u_{i-1})$ exists such that $A = S^{-1}D_x S$, i.e. A is equivalent to $A' = 0$.

Proof. Direct calculations show that C and P are tensors. If A is equivalent to $A' = 0$, then $C' = 0$, and $C = 0$ too. If $C = 0$, find that $\partial_i^2 A = 0$, check the existence of S such that the order of A' in (3) is $i - 1$ or less, then make induction by i down.

Example 1. Take matrix A from 'pathological' ZCR (8), substitute it to (12) and calculate that $C = 0$. Then find matrix S such that (3) changes (8) into $A' = 0$ and $B' =$ constant. The characteristic form (11) of ZCR (1) makes clear that any A with $C = 0$ generates a ZCR for any equation (9), but all such ZCRs are trivial due to theorem 1.

Thus we have seen one of advantages of the characteristic form (11): the characteristic (12) recognizes trivial ZCRs automatically. Ohter advantages of (11), its covariantness and simplicity, allow us to prove the following.

Theorem 2. For every $m \times m$ matrix $A(x, u, u_1, \ldots, u_i)$ of any dimension m and order i, there exists a continual class of matrices B such that (1) is a ZCR for any evolution equation of the continual class

$$u_t=(D_x^r+a_{r-1}D_x^{r-1}+\ldots+a_1D_x+a_0)p+\varphi_1q_1+\ldots+\varphi_sq_s \tag{14}$$

where function $p(x, u, u_1, \ldots, u_j)$, order j and constants $\varphi_1, \ldots, \varphi_s$ are arbitrary, whereas integers r and s and functions $a_0, \ldots, a_{r-1}$ and $q_1, \ldots, q_s$ of $x, u, u_1, \ldots$ are *explicitly determined by A* (as described in the proof).

Proof. Use the characteristic form (11) of ZCR (1). For any given A and corresponding C (12), all matrices P must be found such that ∇P is proportional to C; then (11) is simply a *definition* of all f for represented equations (9), and B is determined by (13). Let $\mathbb{M}$ be the set of all $m\times m$ matrices $M(x, u, u_1, \ldots)$. $\mathbb{M}$ is an m^2-dimensional linear vector space with operations of adding two matrices and multiplying a matrix by any function $d(x, u, u_1, \ldots)$. The operator ∇, $\nabla M=D_xM+[A, M]$, maps $\mathbb{M}$ into $\mathbb{M}$. (Warning: ∇ is *not* a linear operator in $\mathbb{M}$ because $\nabla(dM)=d\nabla M$ if d=constant only.) For the characteristic C (12), consider the sequence of matrices ∇^kC, $k=0, 1, 2, \ldots$. Since $\mathbb{M}$ is finite-dimensional, an integer $r(r\leqslant m^2)$ and functions $c_0, c_1, \ldots, c_{r-1}$ of $x, u, u_1, \ldots$ exist such that $C, \nabla C, \ldots, \nabla^{r-1}C$ are linearly independent and

$$\nabla^rC=\sum_{k=0}^{r-1}c_k\nabla^kC. \tag{15}$$

Under gauge transformations of A, ∇^kC are tensors, therefore c_k are *invariants*. (Note: $r=0$ for trivial A only, when (15) is $C=0$.) Define $\mathbb{C}$, $\mathbb{C}\subset\mathbb{M}$, as the r-dimensional linear vector subspace with the *cyclic basis* $C, \nabla C, \ldots, \nabla^{r-1}C$, $\mathbb{C}$ being invariant with respect to operator ∇. Consider the set $\mathbb{Q}$ of all matrices Q such that $Q\in\mathbb{M}$ and $\nabla Q\in\mathbb{C}$. In $\mathbb{Q}$ mod $\mathbb{C}$, there are not more than s $(s\leqslant m^2-r)$ linearly independent matrices $Q_1, \ldots, Q_s$. (If $s=0$, $\mathbb{Q}=\mathbb{C}$).) Prove that these Q_k can be chosen to satisfy $\nabla Q_k=q_kC$ with some functions $q_k(x, u, u_1, \ldots)$, $k=1, \ldots, s$, and that any Q, $Q\in\mathbb{Q}$, can be decomposed as $Q=Q_0+\varphi_1Q_1+\ldots+\varphi_sQ_s$, where $\varphi_1, \ldots, \varphi_s$ are *constants* and $Q_0\in\mathbb{C}$ (therefore $\mathbb{Q}$ is *not* a linear vector space in that sense as $\mathbb{M}$ and $\mathbb{C}$ are). The *singular basis* $Q_1, \ldots, Q_s$ and *invariants* $q_1, \ldots, q_s$ are determined by A up to transformations $\tilde{Q}_k=\Sigma_{l=1}^s\psi_{kl}Q_l$ and $\tilde{q}_k=\Sigma_{l=1}^s\psi_{kl}q_l$ with any constant non-degenerate $s\times s$ matrix ψ_{kl}. Now return to (11). Since $\nabla P=fC\in\mathbb{C}$, $P\in\mathbb{Q}$. Therefore P can be decomposed as

$$P=\sum_{k=0}^{r-1}p_k\nabla^kC+\sum_{k=1}^{s}\varphi_kQ_k \tag{16}$$

with some functions $p_k(x, u, u_1, \ldots)$ and constants φ_k. Substitute (16) to (11) and find the recursion for p_k

$$p_{k-1}=-D_xp_k-c_kp_{r-1} \qquad k=r-1, r-2, \ldots, 1 \tag{17}$$

and the expression for f

$$f=D_x^rp+D_x^{r-1}(c_{r-1}p)-D_x^{r-2}(c_{r-2}p)+D_x^{r-3}(c_{r-3}p)-\ldots$$
$$+(-1)^{r-1}c_0p+\varphi_1q_1+\ldots+\varphi_sq_s \tag{18}$$

where function $p(x, u, \ldots, u_j)=(-1)^{r-1}p_{r-1}$, order j and constants $\varphi_1, \ldots \varphi_s$ are not fixed. Relation between (18) and (14) is evident. Matrix B is determined by (13) and (16)-(18).

Consequence. For any *finite* set of any finite-order functions $d_1, \ldots, d_m$ of x, u, u_1, $u_2, \ldots$, a continual class of evolution equations exists such that every equation of this class has conserved densities $d_1, \ldots, d_m$. Indeed, if A and B are diagonal matrices, $A = \operatorname{diag}(d_1, \ldots, d_m)$, then ZCR (1) is nothing but m conservation laws for corresponding class (14). Note that $C = \operatorname{diag}(E(d_1), \ldots, E(d_m))$, where E is the Euler operator [8].

Example 2. Take matrix A from (6), calculate $C = \partial_0 A$ and ∇C, and find that $\nabla C = \pm C$, i.e. $r=1$ and $c_0 = \pm 1$ for the cyclic basis. For the singular basis, find that $s=2$, $Q_1 = \operatorname{diag}(1, 0)$, $q_1 = u$, $Q_2 = \operatorname{diag}(1, 1)$, $q_2 = 0$. Then construct f and B, which will be the same as in (7) and (6). Note that invariants of A, i.e. c_0, q_1 and q_2, do not contain parameter α. Prove that α can be removed from A by (3).

Example 3. Take A from (5). Show that C and ∇C are independent but $\nabla^2 C = -u\nabla C$, i.e. $r=2$, $c_0 = 0$, $c_1 = -u$. Then find $s=1$, $Q_1 = \operatorname{diag}(1, 1)$, $q_1 = 0$. Since only traceless matrices were considered in [5], Q_1 is absent from B (5), but this has no effect on f (4) because $q_1 = 0$. Note that invariants c_0, c_1 and q_1 do not contain α. Remove α from A by (3).

It should be stressed that theorem 2 treats A as a *fixed* matrix. Though A depends on parameter α in examples 2 and 3, this dependence is caused only by gauge transformations and has no effect on invariants $c_0, \ldots, c_{r-1}$ and $q_1, \ldots, q_s$, therefore the represented classes are continual as if A contains no parameter at all. On the contrary, if the invariants depend on parameter α, then α is an essential parameter in A. (More strictly, if none of $c_0, \ldots, c_{r-1}$ depend on α, we must check first that α cannot be removed from $q_1, \ldots, q_s$ by new choice of the singular basis.) Theorem 2 is useful in this case too. Indeed, for every fixed value of α in A, theorem 2 provides us with explicit expressions for all admissible f and B in terms of arbitrary function p and constants $\varphi_1, \ldots, \varphi_s$. When α changes, p and $\varphi_1, \ldots, \varphi_s$ may change with α, but the represented equation (9) must be unchanged. The required dependence of p and $\varphi_1, \ldots, \varphi_s$ on α is determined by the condition $\partial f/\partial\alpha = 0$ imposed on f (18). Now, let us see how this condition restricts classes of represented equations when A contains *essential* α. Since the construction of B is evident, we omit it thereafter.

Example 4. Consider

$$A = \begin{pmatrix} 0 & \alpha \\ u & 0 \end{pmatrix} \tag{19}$$

with parameter α. For the cyclic basis, find $r=3$, $c_0 = 2\alpha u_1$, $c_1 = 4\alpha u$, $c_2 = 0$. For the singular basis, find $s=1$, $Q_1 = \operatorname{diag}(1, 1)$, $q_1 = 0$. Then find from (18) that $f = (D_x^3 - 4\alpha u D_x - 2\alpha u_1)p(\alpha, x, u, \ldots, u_j)$. Analyse how p contains α by applying ∂_k to condition $\partial f/\partial\alpha = 0$, $k = j+3, j+2, \ldots, 1$, and find that $p = (\mu x^2 + \nu x + \rho)\alpha^{-1} + \Sigma_{k=0}^{l} h_k \alpha^k$ is necessary, where l is an integer, μ, ν and ρ are constants, and h_k are functions of $x, u, u_1, \ldots$. Then $\partial f/\partial\alpha = 0$ gives the recursion for h_k, $h_k = \frac{1}{4}u^{-1/2}D_x^{-1}u^{-1/2}D_x^3 h_{k+1}$, $k = l-1, l-2, \ldots, 0$, $h_l = \text{constant } u^{-1/2}$, and h_0 determines f:

$$f = -4(2\mu x + \nu)u - 2(\mu x^2 + \nu x + \rho)u_1 + D_x^3 h_0. \tag{20}$$

Thus, (1) with A (19) represents a *discrete* hierarchy. In (20), term $D_x^3 h_0$ gives the hierarchy of the Dym–Kruskal equation $u_t = (u^{-1/2})_{xxx}$ with recursion operator $R = D_x^3 u^{-1/2} D_x^{-1} u^{-1/2}$ (prove that A (19) can be brought by (3) into the Wadati–Konno–

Ichicawa form [2]), but terms with μ, ν and ρ look strange. What is this 'extended Dym–Kruskal hierarchy'? Make the chain of two Miura transformations proposed by Ibragimov [9] for the Dym–Druskal equation: (i) $x=v(y,t)$ and $u=v_y^{-2}$, and then (ii) $w(y,t)=v_y^{-1}v_{yyy}-\frac{3}{2}v_y^{-2}v_{yy}^2$. This chain will connect the whole hierarchy (20) with the Korteweg–de Vries hierarchy (in w), and all 'strange' terms with μ, ν and ρ will disappear at transformation (ii).

Example 5. Consider

$$A=\begin{pmatrix}\alpha & 1\\ \alpha u & -\alpha\end{pmatrix} \tag{21}$$

with parameter α. Find that $r=3$, $c_0=2\alpha u_1$, $c_1=4\alpha(u+\alpha)$, $c_2=0$, $s=1$, $Q_1=\mathrm{diag}(1,1)$, $q_1=0$, $f=[D_x^3-4\alpha(u+\alpha)D_x-2\alpha u_1]p(\alpha,x,u,\ldots,u_j)$. Apply ∂_k to condition $\partial f/\partial\alpha=0$, $k=j+3,\ldots,1$, and show that $p=(\mu x^2+\nu x+\rho)\alpha^{-1}+\Sigma_{k=0}^{l}h_k\alpha^k$ is necessary, where μ, ν, and ρ are constants, l is an integer, and h_k are functions of $x,u,u_1,\ldots$. Then $\partial f/\partial\alpha=0$ leads to

$$f=-4(2\mu x+\nu)u-2(\mu x^2+\nu x+\rho)u_1+D_x^3h_0 \tag{22}$$

$$D_x^3h_1=(4uD_x+2u_1)h_0+4(2\mu x+\nu) \tag{23}$$

$$4D_xh_k+(4uD_x+2u_1)h_{k+1}-D_x^3h_{k+2}=0 \qquad k=l,l-1,\ldots,0 \tag{24}$$

where $h_{l+1}=h_{l+2}=0$ in (24). Find from (24) that $h_l=\sigma=$constant, $h_{l-1}=-\frac{1}{2}\sigma u+\ldots$, $h_{l-2}=\frac{3}{8}\sigma u^2+\ldots$, $h_{l-3}=-\frac{1}{8}\sigma u_2+\ldots$, $h_{l-4}=\frac{5}{16}\sigma uu_2+\ldots$, $h_{l-5}=-\frac{1}{32}\sigma u_4+\ldots$, $h_{l-6}=\frac{7}{64}\sigma uu_4+\ldots$, etc, where only higher-order terms are shown. Then (23) leads to $h_l=\ldots=h_0=0$ and $\mu=\nu=0$. Therefore (1) with A (21) represents *the only* equation, $u_t=$constant u_x.

Example 6. Consider

$$A=\begin{pmatrix}u & v+\alpha\\ 2 & -u\end{pmatrix} \tag{25}$$

where $u=u(x,t)$, $v=v(x,t)$, and α is a parameter. Which systems of the two evolution equations $u_t=f[x,u,v]$ and $v_t=g[x,u,v]$ admit ZCR (1) with A (25)? (Here and below $[x,u,v]$ denotes any finite set of x, u_k and v_k, $k=0,1,2,\ldots$, $u_k=\partial^ku/\partial x^k$, $v_k=\partial^kv/\partial x^k$, $u_0=u$, $v_0=v$.) The whole technique, which we have developed for scalar evolution equations, can be applied to systems with minor modifications. Now the characteristic form (11) of ZCR (1) is

$$fC_u+gC_v=\nabla P \tag{26}$$

where $C_u=\partial A/\partial u$, $C_v=\partial A/\partial v$ and $P[x,u,v]=B$ because A (25) contains no derivatives. *Two* characteristics C_u and C_v and operator ∇ generate cyclic basis C_u, C_v, ∇C_u with *two* closure equations

$$\begin{aligned}\nabla C_v&=-2C_u+2uC_v\\ \nabla^2C_u&=8(v+\alpha)C_u-[2v_1+8u(v+\alpha)]C_v-2u\nabla C_u.\end{aligned} \tag{27}$$

Since coefficients of (27) contain α, this parameter is essential. Singular basis $Q=\mathrm{diag}(1,1)$ has no effect on f and g because $\nabla Q=0$. Decompose P as $P=$

$rC_u + qC_v - p\nabla C_u$, where p, q and r are functions of $[x, u, v]$. Then find from (26) and (27) that $r=(D_x-2u)p$ and

$$f=[D_x^2-2uD_x-2u_1-8(v+\alpha)]p-2q \tag{28}$$

$$g=[2v_1+8u(v+\alpha)]p+(D_x+2u)q \tag{29}$$

where p and q remain arbitrary. Thus, if α is fixed, ZCR (1) with A (25) is admitted by a continual class of systems determined by (28) and (29). If α is free, conditions $\partial f/\partial\alpha = \partial g/\partial\alpha = 0$ must be satisfied. Take (28) as a *definition* of q in terms of f and p, then (29) is $g=-\frac{1}{2}(D_x+2u)f+h$, where $h=[\frac{1}{2}D_x^3-4(w+\alpha)D_x-2w_1]p$, $w=v+\frac{1}{2}(u_1+u^2)$, $w_k=D_x^k w$. Analyse condition $\partial h/\partial\alpha=0$, prove p to be a polynomial in α with coefficients determined recursively, and then find h explicitly: $h=R^n 0$, where $R=D_x^2-8w-4w_1D_x^{-1}$, $n=0, 1, 2, \ldots$, $D_x^{-1}0=\text{constant}$. Thus, h is a function of w_1, $w_2, \ldots$ only, $h=h[w]$, R is the recursion operator of the Korteweg–de Vries equation $w_t=w_3-12ww_1$, therefore $h[w]$ is any constant-coefficient linear superposition of expressions w_1, w_3-12ww_1, $w_5-20ww_3-40w_1w_2+120w^2w_1$, etc. Consequently, ZCR (1) with A (25), which contains *essential* parameter α, is admitted by every system of the continual class

$$u_t=f \qquad v_t=h-\tfrac{1}{2}(D_x+2u)f \tag{30}$$

where function $f(x, u, u_1, u_2, \ldots, v, v_1, v_2, \ldots)$ is *arbitrary*, and $h[w]$ has been determined above.

In conclusion, let us focus our attention on the strange result of example 6: a continual class of systems admits a ZCR with an essential parameter. Moreover, *all* systems (30) possess infinitely many non-trivial conservation laws for the following reason. We find from (30) that w satisfies the equation $w_t=h[w]$ which is a member of the Korteweg–de Vries hierarchy. Since $w_t=h[w]$ is a bi-Hamiltonian equation [8], it has at least the countable set of non-trivial conserved densities: $d[w]=w$, $\frac{1}{2}w^2$, $\frac{1}{2}w_1^2+2w^3$, $\frac{1}{2}w_2^2+10ww_1^2+10w^4, \ldots$ (i.e. d_t are equal to D_x of some functions of w, w_1, $w_2, \ldots$, while non-trivial d are D_x of no functions of w, w_1, $w_2, \ldots$; equation $w_t=\text{constant}\times w_1$ has the continuum of conserved densities $d(w, w, w_2, \ldots)$ with any d). Replacing w by $v+\frac{1}{2}u_1+\frac{1}{2}u^2$ in these $d[w]$, we get infinitely many conserved densities $d[v+\frac{1}{2}u_1+\frac{1}{2}u^2]$ for every system (30) which are all non-trivial (i.e. not equal to D_x of some functions of u, u_1, $u_2, \ldots, v$, v_1, $v_2, \ldots$) because $d[w]$ are non-trivial and $\partial w/\partial v=1$. Can we conclude that all systems (30) are 'integrable'? Undoubtedly, *no*. Though *some* of systems (30) may turn out to be integrable by the inverse scattering transform or exact linearization, we are unable to believe that *every* system of continual class (30) is integrable in some reasonable sense: if one had a technique of integrating (30) for all f, one would be able to integrate *all* evolution equations (9).

Our confidence in that most of systems (30) are non-integrable can be supported by the following result of the Painlevé analysis. Let us consider the five-parameter set of systems

$$\begin{aligned} u_t &= u_3-6u^2u_1-12vu_1+a \\ v_t &= v_3-12vv_1-6u^2v_1-\tfrac{1}{2}D_xa-ua \end{aligned} \tag{31}$$

where $a=\mu u_1+\nu v_1+\rho u^2+\sigma v^2+\tau uv$ with any constants μ, ν, ρ, σ and τ. Systems (31) belong to class (30): f is evident, and $h[w]=w_3-12ww_1$. Let us perform the Painlevé analysis of (31) along the Weiss–Kruskal algorithm [10]. Equations (31) are normal systems with non-characteristic hypersurfaces determined by $\varphi(x,t)=0$ and $\varphi_x\neq 0$ [8], and we put $\varphi_x=1$. Substituting expansions $u=\Sigma_{k=0}^{\infty}\delta_k(t)\varphi^{k+\kappa}$ and $v=\Sigma_{k=0}^{\infty}\varepsilon_k(t)\varphi^{k+\lambda}$ (κ and λ are constants) into (31), we find the following branches to be tested: (i) $\kappa=\lambda=-1$, $\delta_0=\pm 1$, $\varepsilon_0(t)$ is arbitrary, positions of resonances are $k=-1, 0, 1, 3, 4, 5$; (ii) $\kappa=-1$, $\lambda=4$, $\delta_0=\pm 1$, $\varepsilon_0(t)$ is arbitrary, resonances $k=-5, -4, -1, 0, 3, 4$; (iii) $\kappa=5$, $\lambda=-2$, $\delta_0(t)$ is arbitrary, $\varepsilon_0=1$, resonances $k=-7, -5, -1, 0, 4, 6$. Then we find recursion relations for δ_k and ε_k, and check compatibility conditions at resonances. At resonance 1 of branch (i), where arbitrary function $\varepsilon_1(t)$ appears, we get condition $2\mu+\rho-(2\nu+\tau)\varepsilon_0+\sigma\varepsilon_0^2=0$. If this condition is not satisfied for any $\varepsilon_0(t)$, solutions of (31) possess non-dominant logarithmic singularities. Therefore we put $\rho=-2\mu$, $\tau=-2\nu$ and $\sigma=0$. Then we get condition $5\mu\varepsilon_0+2\nu(\varepsilon_1-\varepsilon_0^2)=0$ at resonance 3 of branch (i), where arbitrary function $\delta_3(t)$ appears. We can conclude now that all systems (31) with $a\neq 0$ fail to pass the Painlevé test for integrability because of non-dominant logarithmic branching of their solutions. But system (31) with $a=0$ passes the Painlevé test well: compatibility conditions turn out to be identities at all resonances of all branches. System (31) with $a=0$ is integrable: it is mapped by the Miura transformation $\{w=v+\frac{1}{2}u_1+\frac{1}{2}u^2,\ z=v-\frac{1}{2}u_1+\frac{1}{2}u^2\}$ into the system $\{w_t=w_3-12ww_1,\ z_t=z_3-12zz_1\}$ of two non-coupled Korteweg–de Vries equations (therefore (31) with $a=0$ possesses at least two countable sets of non-trivial conserved densities: $d[v+\frac{1}{2}u_1+\frac{1}{2}u^2]$, common for all systems (30), and additional $d[v-\frac{1}{2}u_1+\frac{1}{2}u^2]$). As for (31) with $a\neq 0$, the non-dominant logarithms are generally considered as a reliable indicator of non-integrability [10, 11].

Fortunately, gauge transformations (3) can shed some light on the strangeness of A (25). Let us look at the following transforming matrix S and resultant matrix A':

$$S=\begin{pmatrix}1 & -\frac{1}{2}u\\ 0 & 1\end{pmatrix}\qquad A'=\begin{pmatrix}0 & v+\frac{1}{2}(u_1+u^2)+a\\ 2 & 0\end{pmatrix}. \tag{32}$$

We see that *two* dependent variables, u and v, have merged into *one*, w, so that *two* equations of system (30) merge into *one*, $w_t=h[w]$, represented by the *equivalent* ZCR. Hence, matrix A (25) contains *effectively* one dependent variable, but not two. This phenomenon requires further investigation.

Acknowledgments

The author is deeply grateful to Dr E V Doktorov whose advice caused this work, to the editors and staff of *J. Phys. A: Math. Gen.* because this paper is the author's tenth publication in this journal, and to the Intellectual Fund of Belarus for support.

References

[1] Novikov S P, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons, the Inverse Scattering Method* (New York: Consultants Bureau)

[2] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)

[3] Dodd R K, Eilbeck J C, Gibbon J D and Morris H C 1982 *Solitons and Nonlinear Wave Equations* (London: Academic)

[4] Cieśliński J, Goldstein P and Sym A 1994 *J. Phys. A: Math. Gen.* 27 1645–64

[5] Rabelo M L and Tenenblat K 1992 *J. Math. Phys.* **33** 537–49
[6] Alexeyev A A and Kudryashov N A 1992 *Teor. Mat. Fiz.* **91** 30–5
[7] Sakovich S Yu 1994 *J. Phys. A: Math. Gen.* **27** L125–9
[8] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[9] Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Dordrecht: Reidel)
[10] Ramani A, Grammaticos B and Bountis T 1989 *Phys. Rep.* **180** 159–245
[11] Clarkson P A and Cosgrove C M 1987 *J. Phys. A: Math. Gen.* **20** 2003–24